A Generalisation of Cramer's Rule

A Project

Submitted by

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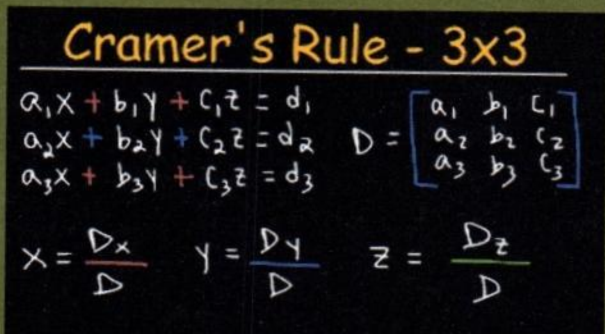

Report

A project on "A Generalisation of Cramer's Rule" was under taken by the students of Department of Mathematics under the guidance of Sri Arabinda Pandab, HOD Mathematics. It took two months (Feb & March 2020) to carried out the project. Cramer's rule is an explicit formula for the solution of a system of linear equations with many unknowns, valid whenever the system has a unique solution. It expresses the solutions in terms of the determinants of the (square) coefficient matrix and of matrices obtained from it by replacing one column by the column vector of right hand sided of the equations. It is named after Gabriel Cramer (1704-1752) who published the rule obtaining unique solution of system of linear equations with arbitrary number of unknowns same as the number of equations. The students found Generalization of Cramer's Rule for Finding Solution of System Ax = b, where A is a Square Matrix, Generalization of Cramer's Rule for Finding Solution of System Ax = b, where A is a Rectangular Matrix, Variational method to obtain solutions of the systems Ax=b, where A is a rectangular matrix.

Finally, the project was completed and submitted on 10th March 2020.

10/3/20

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Contents

Chapter 1

INTRODUCTION

1.1 Introduction

Cramer's rule is an explicit formula for the solution of a system of linear equations with many unknowns, valid whenever the system has a unique solution. It express the solutions in terms of the determinants of the (square) coefficient matrix and of matrices obtained from it by replacing one column by the column vector of right hand sided of the equations. It is named after Gabriel Cramer(1704-1752)who published the rule obtaining unique solution of system of linear equations with arbitrary number of unknowns same as the number of equations.

Consider a system of n linear equations with n unknowns, represented in matrix multiplication form as

$$Ax = b. \tag{1.1}$$

If A is $n \times n$ matrix with $det(A) \neq 0$, then the solution of the system (1.1) is given by the formula:

$$x_i = \frac{det((A)_i)}{det(A)},\ i = 1, 2, 3, ..., n,$$

where $(A)_i$ is the matrix obtained by replacing the entries in the ith column of A by the entries

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Consider a system of n linear equations with m unknowns, represented in matrix multiplication form as

$$Ax = b, x \in R^m, b \in R^n, m \geq n, \tag{1.2}$$

where $A = (a_{j,i})_{n\times m}$ is a $n \times m$ real matrix. i.e.,

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,m}x_m = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + ... + a_{n,m}x_m = b_n \end{cases}$$

We can apply Gauss elimination method to find some solutions of this systems and this method is a systematic procedure for solving systems like (1.2); it is based on the idea of reducing the augmented matrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & b_n \end{bmatrix}$$

to the form that is simple enough such that the system of equations can be solved by inspection. But, in general there is no formula for the solutions of (1.2) in terms of determinants if $m \neq n$.
When m=n and det(A) $\neq 0$, the system (1.2) admits only one solution given by x=$A^{-1}b$, and from here we can deduce the well known Cramer's rule.

Hugo Leiva has found two formulas for the solutions of system of linear equations, where the number of equations and number of unknowns are not equal and this formula coincides with the Cramer's rule when n=m. This generalized Cramer's rule, has applications in differential geometry, computing derivatives implicitly, integer programming, ordinary differential equations etc.

1.2 Preliminaries:

Definition 1.2.1. (**Inner Product Space**)

Let x be a vector space. Define $\langle\ ,\ \rangle : X \times x \longrightarrow C$. $\langle\ ,\ \rangle$ is said to be inner product space if it satisfies the following properties,

(i) $\langle x + y,\ z\rangle = \langle x,\ z\rangle + \langle y,\ z\rangle$

(ii) $\langle \alpha x,\ y\rangle = \alpha\langle x,\ y\rangle$

(iii) $\langle x,\ y\rangle = \overline{\langle y,\ x\rangle}$

(iv) $\langle x,\ x\rangle \geq 0$

$\langle x,\ x\rangle = 0$ iff $x = 0$.

Definition 1.2.2. (Hilbert space)

An inner product space is said to be a Hilbert space, if it is complete with respect to metric induced from the inner product space.

Definition 1.2.3. (Adjoint Operator)

Let $T : H_1 \longrightarrow H_2$ be a bounded linear operator, where $H_1 and H_2$ are Hilbert spaces .Then the Hilbert -adjoint operator T^* of T is the operator

$$T^* : H_2 \longrightarrow H_1$$

such that for all $x \in H_1$ and $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

1.3 Chapterwise Summary

The dissertation consists of three chapters.

This dissertation is to study about the generalization of Cramer's rule.

Chapter one is all about the introduction, preliminaries and summary of this dissertation. More precisely, it contains some formal definitions and results which are essential for a clear understanding of the subsequent chapters.

Chapter two consist of two sections. This chapter deals with the study of the generalization of Cramer's rule. The first section gives the formula for finding solution of system $Ax = b$, where A is a square matrix. In the second section, we will prove some theorems and discuss some examples regrading to generalization of Cramer's rule for finding solution of system $Ax = b$, where A is a rectangular matrix.

Chapter three deals with the study of a variational method to obtain solutions of rectangular matrix.

Chapter 2

A GENERALIZATION OF CRAMER'S RULE

In this chapter we will discuss about the generalization of Cramer's rule for finding solution of system $Ax = b$, where A is a square matrix and rectangular matrix. Also, we will prove some theorems and discuss some examples regrading to generalization of Cramer's rule.

2.1 Generalization of Cramer's Rule for Finding Solution of System $Ax = b$, where A is a Square Matrix

A simple and interested generalization of Cramer Rule is done by Prof. Dr. Sylvan Burgstahler from University of Minnesota, Duluth, where he taught for 20 years. This result is given by the following Theorem:

Theorem 2.1.1. *[1]*(**Burgstahler 1983**) *If the system of equations*

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + ... + a_{n,n}x_n = b_n \end{cases} \tag{2.1}$$

has unique solution $x_1, x_2, ..., x_n$, then for all $\lambda_i \in R, i = 1, 2, ..., n$ one has

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = \frac{\begin{vmatrix} a_{1,1} + \lambda_1 b_1 & a_{1,2} + \lambda_2 b_1 & \cdots & a_{1,n} + \lambda_n b_1 \\ a_{2,1} + \lambda_1 b_2 & a_{2,2} + \lambda_2 b_2 & \cdots & a_{2,n} + \lambda_n b_2 \\ \vdots & \vdots & & \vdots \\ a_{n,1} + \lambda_1 b_n & a_{n,2} + \lambda_2 b_n & \cdots & a_{n,n} + \lambda_n b_n \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}} - 1. \tag{2.2}$$

Proof. Let $N = \begin{vmatrix} a_{1,1}+\lambda_1 b_1 & a_{1,2}+\lambda_2 b_1 & \cdots & a_{1,n}+\lambda_n b_1 \\ a_{2,1}+\lambda_1 b_2 & a_{2,2}+\lambda_2 b_2 & \cdots & a_{2,n}+\lambda_n b_2 \\ \vdots & \vdots & & \vdots \\ a_{n,1}+\lambda_1 b_n & a_{n,2}+\lambda_2 b_n & \cdots & a_{n,n}+\lambda_n b_n \end{vmatrix}$.
By the rules for adding determinants,

$$N = \begin{vmatrix} a_{1,1}+\lambda_1 b_1 & \cdots & a_{1,n-1}+\lambda_{n-1} b_1 & a_{1,n} \\ a_{2,1}+\lambda_1 b_2 & \cdots & a_{2,n-1}+\lambda_{n-1} b_2 & a_{2,n} \\ \vdots & & \vdots & \vdots \\ a_{n,1}+\lambda_1 b_n & \cdots & a_{n,n-1}+\lambda_{n-1} b_n & a_{n,n} \end{vmatrix} + \begin{vmatrix} a_{1,1}+\lambda_1 b_1 & \cdots & a_{1,n-1}+\lambda_{n-1} b_1 & \lambda_n b_1 \\ a_{2,1}+\lambda_1 b_2 & \cdots & a_{2,n-1}+\lambda_{n-1} b_2 & \lambda_n b_2 \\ \vdots & & \vdots & \vdots \\ a_{n,1}+\lambda_1 b_n & \cdots & a_{n,n-1}+\lambda_{n-1} b_n & \lambda_n b_n \end{vmatrix}$$

$$= \begin{vmatrix} a_{1,1}+\lambda_1 b_1 & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1}+\lambda_1 b_2 & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & & \vdots & \vdots \\ a_{n,1}+\lambda_1 b_n & \cdots & a_{n,n-1} & a_{n,n} \end{vmatrix} + \begin{vmatrix} a_{1,1}+\lambda_1 b_1 & \cdots & \lambda_{n-1} b_1 & a_{1,n} \\ a_{2,1}+\lambda_1 b_2 & \cdots & \lambda_{n-1} b_2 & a_{2,n} \\ \vdots & & \vdots & \vdots \\ a_{n,1}+\lambda_1 b_n & \cdots & \lambda_{n-1} b_n & a_{n,n} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{1,1}+\lambda_1 b_1 & \cdots & a_{1,n-1} & \lambda_n b_1 \\ a_{2,1}+\lambda_1 b_2 & \cdots & a_{2,n-1} & \lambda_n b_2 \\ \vdots & & \vdots & \vdots \\ a_{n,1}+\lambda_1 b_n & \cdots & a_{n,n-1} & \lambda_n b_n \end{vmatrix} + \begin{vmatrix} a_{1,1}+\lambda_1 b_1 & \cdots & \lambda_{n-1} b_1 & \lambda_n b_1 \\ a_{2,1}+\lambda_1 b_2 & \cdots & \lambda_{n-1} b_2 & \lambda_n b_2 \\ \vdots & & \vdots & \vdots \\ a_{n,1}+\lambda_1 b_n & \cdots & \lambda_{n-1} b_n & \lambda_n b_n \end{vmatrix}.$$

From the last determinant taking common λ_{n-1} from $(n-1)$th column and taking common λ_n from nth column, we see that last determinant vanishes because it contains two identical columns.

Continuing in this way, we get

$$N = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix} + \begin{vmatrix} \lambda_1 b_1 & a_{1,2} & \cdots & a_{1,n} \\ \lambda_1 b_2 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ \lambda_1 b_n & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}$$
$$+ \cdots + \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & \lambda_n b_1 \\ a_{2,1} & a_{2,2} & \cdots & \lambda_n b_2 \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & \lambda_n b_n \end{vmatrix}$$

Dividing the first determinant in both side of the above equation, we get

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = \frac{\begin{vmatrix} a_{1,1}+\lambda_1 b_1 & a_{1,2}+\lambda_2 b_1 & \cdots & a_{1,n}+\lambda_n b_1 \\ a_{2,1}+\lambda_1 b_2 & a_{2,2}+\lambda_2 b_2 & \cdots & a_{2,n}+\lambda_n b_2 \\ \vdots & \vdots & & \vdots \\ a_{n,1}+\lambda_1 b_n & a_{n,2}+\lambda_2 b_n & \cdots & a_{n,n}+\lambda_n b_n \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}} - 1$$

□

2.2 Generalization of Cramer's Rule for Finding Solution of System $Ax = b$, where A is a Rectangular Matrix

Consider a system of n linear equations with m unknowns, represented in matrix multiplication form as

$$Ax = b, x \in R^m, b \in R^n, m \geq n, \tag{2.3}$$

where $A = (a_{j,i})_{n\times m}$ is a $n \times m$ real matrix. i.e.,

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,m}x_m = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + ... + a_{n,m}x_m = b_n. \end{cases} \tag{2.4}$$

Now if we defined the column vectors

$$l_1 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix}, l_2 = \begin{bmatrix} a_{2,1} \\ a_{2,2} \\ \vdots \\ a_{2,m} \end{bmatrix}, \ldots, l_n = \begin{bmatrix} a_{n,1} \\ a_{n,2} \\ \vdots \\ a_{n,m} \end{bmatrix},$$

then the system (2.4) also can be written as follows:

$$\langle l_i, x \rangle = b_i, i = 1, 2, ..., n, \tag{2.5}$$

where $\langle ., . \rangle$ denotes the inner product in R^m and A is $n \times m$ real matrix. A necessary and sufficient conditon is established for finding out the solution of the system of equation (2.3) in the following theorem.

Theorem 2.2.1. *For all $b \in R^n$, the system (2.3) is solvable if and only if ,*

$$det(AA^*) \neq 0. \tag{2.6}$$

Moreover, one solution for this equation is given by the following formula:

$$x = A^*(AA^*)^{-1}b, \tag{2.7}$$

where A^* is the transpose of A (or the conjugate transpose of A in the complex case).
Also, this solution coincides with the Cramer formula when $n = m$. Infact, this formula is given as follows:

$$x_i = \sum_{j=1}^{n} a_{j,i} = \frac{det((AA^*)_j)}{det(AA^*)}, i = 1, 2, ..., m, \tag{2.8}$$

where $(AA^*)_j$ is the matrix obtained by replacing the entries in the jth column of AA^* by the entries in the matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

In addition, this solution has minimum norm,i.e.,

$$\|x\| = inf\{\|w\| : Aw = b, w \in R^m\} \tag{2.9}$$

and $\|x\| = \|w\|$ *with* $Aw = b \Leftrightarrow x = w$.

Proof. Let us denote $\langle x, y\rangle$ be the Euclidian inner product in R^k and the associated norm by $\|x\| = \sqrt{\langle x, y\rangle}$. Also, we shall use some ideas from [3] and the following result from [2].
Lemma : Let W and Z be Hilbert spaces, $G \in L(W, Z)$ and $G^* \in L(Z, W)$ the adjoint operator, then the following statements holds,
[(*i*)] Range(G) $= Z \Leftrightarrow \exists\gamma > 0$ such that

$$\|G^*z\|_w \geqslant \gamma\|z\|_z, \quad z \in Z.$$

[(*ii*)] $\overline{\text{Range}(G)} = Z \Leftrightarrow ker(G^*) = \{0\} \Leftrightarrow G^*$ is $1-1$.

The matrix A is a linear operator $A : R^n \rightarrow R^m$. Therefore $A \in L(R^m, R^n)$ and its adjoints operator A^* is the transpose of A and $A^* : R^n \rightarrow R^m$.
Then ,system (2.3) is solvable for all $b \in R^n$, if and only if the operator A is surjective. Hence, from the above lemma there exists $\gamma > 0$ such that

$$\|A^*z\|_{R^m} \geqslant \gamma\|z\|_{R^n}, z \in R^n$$

i.e.,

$$\|A^*z\|^2_{R^m} \geqslant \gamma^2\|z\|^2_{R^n}, z \in R^n$$

i.e.,

$$\langle A^*z, A^*z\rangle \geqslant \gamma^2\|z\|^2_{R^n}, z \in R^n$$

i.e,

$$\langle AA^*z, z\rangle \geqslant \gamma^2\|z\|^2_{R^n}, z \in R^n.$$

This implies that AA^* is one to one. Since AA^* is a $n \times n$ matrix, then $\det(AA^*) \neq 0$.
Now suppose that $\det(AA^*) \neq 0$. Then $(AA^*)^{-1}$ exists and given $b \in IR^n$.

Here $x = (AA^*)^{-1}b$ is a solution of $Az = b$.
Now, since $z = (AA^*)^{-1}b$ is the only solution of the equation

$$(AA^*)w = b$$

, then from the Cramer's rule we obtained that:

$$z_1 = \frac{det((AA^*)_1)}{det(AA^*)},\ z_2 = \frac{det((AA^*)_2)}{det(AA^*)},\ ...,\ z_n = \frac{det((AA^*)_n)}{det(AA^*)},$$

where (AA^*) is the matrix obtained by replacing the entries in the ith column of AA^* by the entries in the matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then, the solution $x = A^*(AA^*)^{-1}b$ of (2.3) can be written as follows

$$x = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & b_n \end{bmatrix} \begin{bmatrix} \frac{det((AA^*)_1)}{det(AA^*)} \\ \frac{det((AA^*)_2)}{det(AA^*)} \\ \vdots \\ \frac{det((AA^*)_n)}{det(AA_*)} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{j,1}\frac{det((AA^*)_1)}{det(AA^*)} \\ \sum_{j=1}^{n} a_{j,2}\frac{det((AA^*)_2)}{det(AA^*)} \\ \vdots \\ \sum_{j=1}^{n} a_{j,m}\frac{det((AA^*)_n)}{det(AA^*)} \end{bmatrix}.$$

Hence,

$$x_i = \sum_{j=1}^{n} a_{j,i}\frac{det((AA^*)_j)}{det(AA^*)},\quad i = 1, 2, ..., m.$$

Now, we have to show that this solution has minimum norm. Let us consider w in R^m such that $Aw = b$ and

$$\|w\|^2 = \|x + (w - x)\|^2 = \|x\|^2 + 2Re\langle x, w - x\rangle + \|w - x\|^2$$

on the other hand,

$$\langle x, w-x\rangle = \langle A^*(AA^*)^{-1}b, w-x\rangle = \langle (AA^*)^{-1}b, Aw-Ax\rangle = \langle (AA^*)^{-1}b, b-b\rangle = 0.$$

Hence, $\|w\|^2 - \|x\|^2 = \|w - x\|^2 \geqslant 0$.
Therefore, $\|x\| \leqslant \|w\|$, and $\|x\| = \|w\|$ if $x = w$. □

The solution of the system of equations (2.3) as written in (2.7) can also be expressed in different form, which is proved in the following theorem.

Theorem 2.2.2. The solution of (2.3) **is given by**

$$x_i = \frac{\begin{vmatrix} \|l_1\|^2 + a_{1i}b_1 & \langle l_1, l_2\rangle + a_{2i}b_1 & \cdots & \langle l_1, l_n\rangle + a_{ni}b_1 \\ \langle l_2, l_n\rangle + a_{1i}b_2 & \|l_2\|^2 + a_{2i}b_2 & \cdots & \langle l_1, l_n\rangle + a_{ni}b_2 \\ \vdots & \vdots & & \vdots \\ \langle l_1, l_n\rangle + a_{1i}b_2 & \langle l_1, l_n\rangle + a_{2i}b_n & \cdots & \|l_n\|^2 + a_{ni}b_n \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2\rangle & \cdots & \langle l_1, l_n\rangle \\ \langle l_2, l_1\rangle & \|l_2\|^2 & \cdots & \langle l_2, l_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle l_n, l_1\rangle & \langle l_n, l_2\rangle & \cdots & \|l_n\|^2 \end{vmatrix}} - 1, \; i = 1, 2, 3, ..., m. \tag{2.10}$$

In the following, we will verify this theorem for n=2.
Let us consider, the following system of equations,

$$a_{1,1}x_1 + a_{1,2}x_2 = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 = b_2.$$

Here $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, $A^* = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{bmatrix}$.

$$AA^* = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{bmatrix} = \begin{bmatrix} \| l_1 \|^2 & \langle l_1, l_2\rangle \\ \langle l_2, l_1\rangle & \| l_2 \|^2 \end{bmatrix}.$$

Now, det $(AA^*) = \|l_1\|^2\|l_2\|^2 - |\langle l_1, l_2\rangle|^2$.
As per the solution in (2.7), we have

$$x_i = \sum_{j=1}^{n} a_{j,i} = \frac{det((AA^*)_j)}{det(AA^*)}, \; i = 1, 2, ..., m.$$

$$=a_{1i}\frac{det(AA^*)_1}{det(AA^*)}+a_{2i}\frac{det(AA^*)_2}{det(AA^*)}$$

$$= a_{1i}\frac{\begin{vmatrix} b_1 & \langle l_1, l_2\rangle \\ b_2 & \|l_2\|^2 \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2\rangle \\ \langle l_2, l_1\rangle & \|l_2\|^2 \end{vmatrix}} + \frac{\begin{vmatrix} \|l_1\|^2 & b_1 \\ \langle l_2, l_1\rangle & b_2 \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2\rangle \\ \langle l_2, l_1\rangle & \|l_2\| \end{vmatrix}}.$$

For i=1

$$x_1 = \frac{a_{11}b_1\|l_2\|^2 - a_{11}b_2\langle l_1, l_2\rangle + a_{21}b_2\|l_1\|^2 - a_{21}b_1\langle l_2, l_1\rangle}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2\rangle \\ \langle l_2, l_1\rangle & \|l_2\|^* \end{vmatrix}}. \tag{2.11}$$

As per the solution in (2.10),

$$\begin{aligned} x_1 &= \frac{\begin{vmatrix} \|l_1\|^2 + a_{11}b_1 & \langle l_1, l_2\rangle + a_{21}b_1 \\ \langle l_2, l_n\rangle + a_{11}b_2 & \|l_2\|^2 + a_{21}b_2 \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2\rangle \\ \langle l_2, l_1\rangle & \|l_2\|^2 \end{vmatrix}} - 1 \\ &= \frac{a_{11}b_1\|l_2\|^2 - a_{11}b_2\langle l_1, l_2\rangle + a_{21}b_2\|l_1\|^2 - a_{21}b_1\langle l_2, l_1\rangle}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2\rangle \\ \langle l_2, l_1\rangle & \|l_2\|^* \end{vmatrix}}. \end{aligned} \tag{2.12}$$

We get the same solution for x_1 in equation (2.11) and(2.16) obtained by the two formulas as in (2.7) and (2.10).

Theorem 2.2.3. *The system* (2.3) *is solvable for each* $b \in R^n$, *if and only if the set of vectors* $l_1, l_2, ..., l_n$ *formed by the rows of the matrix* A *is linearly independent in* R^n. *Moreover, a solution for the system* (2.3) *is given by the following formula :*

$$x_i = v_{1i}\|v_1\|^2 b_1 + v_{2i}\|v_1\|^2(b_2 - < l_1, v_1 >)\|v_1\|^2 c_1), \tag{2.13}$$

$$+ ... + v_{ni}(b_n - \sum_{i=1}^{n-1} < l_1, v_i > c_i),\ i = 1, 2, ..., m,$$

Where the set of vectors $v_1, v_2, ..., v_n$ are obtained from $\{l_1, l_2, ..., l_n\}$ by the Gram Schmidt process and the numbers $c_1, c_2, ..., c_n$ are given by

$$\begin{cases} c_1 = b_1 \\ c_2 = b_2 - < l_2, v_1 > \|v_1\|^2 c_1 \\ c_3 = b_3 - < l_3, v_1 > \|v_1\|^2 c_1 - < l_3, v_2 > \|v_2\|^2 c_2 \\ \vdots \\ c_n = b_n - \sum_{i=1}^{n-1} < l_1, v_i > c_i \end{cases} \tag{2.14}$$

and $v_i = [v_{i1}, v_{i2}, v_{i3} ..., v_{im}]^T$, $i = 1, 2, ..., n$.

Proof. Suppose the system is solvable for all $b \in R^n$. Now, assume the existence of real numbers c^i, $i = 1, 2, ..., n$ such that

$$c_1 l_1 + c_2 l_2 + c_3 l_3 + ... + c_n l_n = 0.$$

Then, their exist x $\in R^m$ such that

$$a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,m}x_m = c_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,m}x_m = c_2$$

$$\vdots$$

$$a_{n,1}x_1 + a_{n,2}x_2 + ... + a_{n,m}x_m = c_n.$$

In other words,

$$< l_i, x >= c_i,\ i = 1, 2, ..., n.$$

Hence,

$$< c_i l_i, x >= c_i^2,\ i = 1, 2, ..., n.$$

So,

$$< c_1 l_1 + c_2 l_2 + c_3 l_3 + ... c_n l_n, x >= c_1^2 + c_2^2 + c_3^2 + ... + c_n^2 = 0.$$

Therefore, $c_1 = c_2 = c_3 = ... = c_n = 0$, which proves the independence of $\{l_1, l_2, ..., l_n\}$. Now, suppose that the set $\{l_1, l_2, ..., l_n\}$ is linearly independent in R^m. Using the Gram-Schmidt process we can find a set $\{v_1, v_2, ..., v_n\}$ of orthogonal vectors in IR^m given by the formula:

$$\begin{cases} v_1 = l_1, \\ v_2 = l_2 - \dfrac{< l_2, v_1 >}{\|v_1\|^2} v_1, \\ v_3 = l_3 - \dfrac{< l_3, v_1 >}{\|v_1\|^2} v_1 - \dfrac{< l_3, v_2 >}{\|v_2\|^2} v_2, \\ \vdots \\ v_n = l_n - \sum_{i=1}^{n-1} \dfrac{< l_n, v_i >}{\|v_i\|^2} v_i. \end{cases} \tag{2.15}$$

Then, system (2.3) will be equivalent to the following system:

$$\begin{cases} < v_1, x >= c_1 \\ < v_2, x >= c_2 \\ < v_3, x >= c_3 \\ \vdots \\ < v_n, x >= c_n \end{cases} \tag{2.16}$$

where

$$\begin{cases} c_1 = b_1 \\ c_2 = b_2 - \dfrac{< l_2, v_1 >}{\|v_1\|^2} c_1 \\ c_3 = b_3 - \dfrac{< l_3, v_1 >}{\|v_1\|^2} c_1 - \dfrac{< l_3, v_2 >}{\|v_2\|^2} c_2 \\ \vdots \\ c_n = b_n - \sum_{i=1}^{n-1} \dfrac{< l_n, v_i >}{\|v_i\|^2} c_i. \end{cases} \tag{2.17}$$

If we denote the vectors $v_i's$ by

$$v_i = \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \\ \vdots \\ v_{in} \end{bmatrix}, i = 1, 2..., n,$$

and the $n \times m$ matrix γ by

$$\gamma = \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,m} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,m} \\ \vdots & \vdots & & \vdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,m} \end{bmatrix}$$

using Theorem (2.2.1) we obtain that system (2.16) has solution for all $C \in R^n$, if and only if, $det(\gamma\gamma^* \neq 0)$. But,

$$\gamma\gamma^* = \begin{bmatrix} \| v_1 \|^2 & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \| v_2 \|^2 & \cdots & \langle v_2, v_n \rangle \\ \vdots & \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \cdots & \| v_n \|^2 \end{bmatrix} = \begin{bmatrix} \| v_1 \|^2 & 0 & \cdots & 0 \\ 0 & \| v_2 \|^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \| v_n \|^2 \end{bmatrix}$$

So $\det(\gamma\gamma^*) = \| v_1 \|^2 \| v_2 \|^2 \cdots \| v_n \|^2 \neq 0$.
Using the formula (2.7), we get,

$$x_i = v_{1i}\|v_1\|^2 b_1 + v_{2i}\|v_1\|^2(b_2 - < l_1, v_1 >)\|v_1\|^2 c_1), + ... + v_{ni}(b_n - \sum_{i=1}^{n-1} < l_1, v_i > c_i),$$

$i = 1, 2, ..., m$.

□

Example 2.2.4.
Consider the following particular case of system (1.2)

$$a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,m}x_m = b. \tag{2.18}$$

In this case $n = 1$ and $A = [a_{1,1}, a_{1,2}, ..., a_{1,m}]$.
Then, let us define the column vector is

$$l_i = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ a_{1,3} \\ \vdots \\ a_{1,n} \end{bmatrix},$$

$$AA^* = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \end{bmatrix} \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ a_{1,3} \\ \vdots \\ a_{1,n} \end{bmatrix} = \| l_1 \|^2 .$$

Then, $(AA^*)^{-1}b = b \| l_1 \|^{-2}$ and

$$x = A^*(AA^*)^{-1}b = \begin{bmatrix} a_{1,1}b \| l_1 \|^{-2} \\ a_{1,2}b \| l_1 \|^{-2} \\ \vdots \\ a_{1,n}b \| l_1 \|^{-2} \end{bmatrix} .$$

Therefore, the solution of the system (2.18) is given by:

$$\begin{aligned} x_i &= \sum_{j=1}^{n} a_{j,i} = \frac{det((AA^*)_j)}{det(AA^*)},\ i = 1, 2, ..., m. \\ &= \frac{a_{1i}b}{\| l_1 \|^2} = \frac{a_{1i}b}{\sum_{j=1}^{m} a_{1j}^2},\ i = 1, 2, ..., m. \end{aligned} \tag{2.19}$$

Example 2.2.5.

Consider the following particular case of system (1.2)

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,m}x_m = b_2. \end{cases} \tag{2.20}$$

In this case n=2 and

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots a_{2,m} \end{bmatrix} .$$

Then, if we define the column vectors

$$l_1 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ a_{1,3} \\ \vdots \\ a_{1,m} \end{bmatrix},\ l_2 = \begin{bmatrix} a_{2,1} \\ a_{2,2} \\ a_{2,3} \\ \vdots \\ a_{2,m} \end{bmatrix},$$

then

$$AA^* = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots a_{2,m} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \\ \vdots & \vdots \\ a_{1,m} & a_{2,m} \end{bmatrix}$$

$$= \begin{bmatrix} \| l_1 \|^2 & \langle l_1, l_2 \rangle \\ \langle l_2, l_1 \rangle & \| l_2 \|^2 \end{bmatrix}.$$

Hence, from the formula (2.8) we obtain that:

$$\begin{bmatrix} x_{1,1} \\ x_{1,2} \\ x_{1,3} \\ \vdots \\ x_{1,m} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \\ \vdots & \vdots \\ a_{1,m} & a_{2,m} \end{bmatrix} \begin{bmatrix} \dfrac{det((AA^*)_1)}{det(AA^*)} \\ \\ \dfrac{det((AA^*)_2)}{det(AA^*)} \end{bmatrix}.$$

Therefore, a solution of the system (2.20) is given by:

$$x_1 = a_{11} \left[\frac{det((AA^*)_1)}{det(AA^*)} \right] + a_{21} \left[\frac{det((AA^*)_2)}{det(AA^*)} \right]$$

i.e,

$$x_1 = a_{11} \frac{\begin{vmatrix} b_1 & \langle l_1, l_2 \rangle \\ b_2 & \|l_2\|^2 \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2 \rangle \\ \langle l_2, l_1 \rangle & \|l_2\|^2 \end{vmatrix}} + a_{21} \frac{\begin{vmatrix} \|l_1\|^2 & b_1 \\ \langle l_2, l_1 \rangle & b_2 \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & \langle l_1, l_2 \rangle \\ \langle l_2, l_1 \rangle & \|l_2\|^* \end{vmatrix}}$$

i.e.,

$$= a_{11} \frac{b_1 \|l_2\|^2 - b_2 \langle l_1, l_2 \rangle}{\|l_1\|^2 \|l_2\|^2 - |\langle l_1, l_2 \rangle|^2} + a_{21} \frac{b_2 \|l_1\|^2 - b_1 \langle l_2, l_1 \rangle}{\|l_1\|^2 \|l_2\|^2 - |\langle l_1, l_2 \rangle|^2} \tag{2.21}$$

$$x_2 = a_{12} \frac{b_1 \|l_2\|^2 - b_2 \langle l_1, l_2 \rangle}{\|l_1\|^2 \|l_2\|^2 - |\langle l_1, l_2 \rangle|^2} + a_{22} \frac{b_2 \|l_1\|^2 - b_1 \langle l_2, l_1 \rangle}{\|l_1\|^2 \|l_2\|^2 - |\langle l_1, l_2 \rangle|^2} \tag{2.22}$$

$$\vdots$$

$$x_m = a_{1m}\frac{b_1\|l_2\|^2 - b_2\langle l_1, l_2\rangle}{\|l_1\|^2\|l_2\|^2 - |\langle l_1, l_2\rangle|^2} + a_{2m}\frac{b_2\|l_1\|^2 - b_1\langle l_2, l_1\rangle}{\|l_1\|^2\|l_2\|^2 - |\langle l_1, l_2\rangle|^2} \tag{2.23}$$

Now, we shall apply the foregoing formula to find the solution of the following system

$$\begin{cases} x_1 + x_2 = 1, \\ -x_1 + x_2 + x_3 = -1. \end{cases} \tag{2.24}$$

If we define the column vectors

$$l_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, l_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Here, $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $A^* = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Now, $\det(AA^*) = \|l_1\|^2\|l_2\|^2 - |\langle l_1, l_2\rangle|^2 = \|l_1\|^2\|l_2\|^2 = 6$

$$x_1 = \frac{5}{6}, x_2 = \frac{1}{6}, x_3 = \frac{-2}{6}.$$

Example 2.2.6.

Consider the following general case of system (1.2)

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,m}x_m = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,m}x_m = b_n \end{cases}. \tag{2.25}$$

Then, if $\{l_1, l_2, ..., l_n\}$ of orthogonal sets in R^m , we get

$$(AA^*) = \begin{bmatrix} \| l_1 \|^2 & 0 & \cdots & 0 \\ 0 & \| l_2 \|^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \| l_n \|^2 \end{bmatrix}$$

and the solution of the system (2.3) is very simple and given by:

$$x_i = \sum_{j=1}^{n} a_{j,i} b_j \|l_2\|^- 2, i = 1, 2, ..., m. \qquad (2.26)$$

Now, we shall apply the above formula to find the solution of the following system:

$$\begin{cases} -x_1 - x_2 + x_3 + x_4 = 1, \\ -x_1 + x_2 - x_3 + x_4 = 1, \\ x_1 - x_2 - x_3 + x_4 = 1. \end{cases} \qquad (2.27)$$

If we define the column vectors

$$l_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, l_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, l_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Here, $A = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ and $A^* = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

Then, $\{l_1, l_2, l_3\}$is an orthogonal set in R^3 and the solution of the system is given by:

$$x_1 = \frac{-1}{4}, \; x_2 = \frac{-1}{4}, \; x_3 = \frac{-1}{4}, \; x_1 = \frac{3}{4}.$$

Chapter 3

VARIATIONAL METHOD TO OBTAIN SOLUTIONS SYSTEM $Ax = b$, *WHERE* A *IS A RECTANGULAR MATRIX*

In the previous chapter we find out a formula for one solution of the system (2.3) which has minimum norm. But it is not the only way to find the solution of this equation. Now, in this chapter we will find a variational method to obtain solution of (2.3) of minimum norm.

Consider the quadratic functional $j : R^n \to R$

$$j(\xi) = \frac{1}{2}\| \in A^*\xi\| \in^2 - \langle b, \xi\rangle, \quad \forall\xi \in R^n. \tag{3.1}$$

Proposition 3.0.1. *For a given* $b \in R^n$ *the equation has a solution* $x \in R^m$ *if and only if,*

$$\langle x, A^*\xi\rangle - \langle b, \xi\rangle = 0, \quad \forall\xi \in R^n. \tag{3.2}$$

Lemma 3.0.2. Suppose the quadratic functional j has a minimizer $\xi_b \in R^n$. Then,

$$x_b = A^*\xi_b \tag{3.3}$$

is a solution of (2.3).

Proof. We have

$$j(\xi) = \frac{1}{2}\|A^*\xi\|^2 - \langle b, \xi\rangle, \quad \forall\xi\epsilon \in R^n$$

$$= \frac{1}{2}\langle AA^*\xi, \xi\rangle - \langle b, \xi\rangle, \quad \forall\xi \in R^n.$$

Then, if ξ_b is a point where j achieves its minimum value, we obtain that:

$$\frac{d}{d\xi}\{j\}(\xi_b) = AA^*\xi_b - b = 0.$$

So, $AA^*\xi_b = b$ *and* $X_b = A^*\xi_b$ is a solution of (2.3).

□

Theorem 3.0.3. *The system* (2.3) *is solvable if and only if the quadratic functional* j *defined by* (3.1) *has a minimum value for all* $b \in R^n$.

Proof. Suppose $\det(AA^*) \neq 0$ is solvable. Then, the matrix A is an operator from R^m to R^n is surjective.
Hence, from the above lemma there exist $\gamma > 0$ such that

$$\|A^*\xi\|^2 \geqslant \gamma^2\|\xi\|^2, \quad \xi \in R^n.$$

Then,

$$j(\xi) \geqslant \frac{\gamma^2}{2}\|\xi\|^2 - \|\|b\|\xi\|, \quad \xi \in R^n.$$

Therefore,

$$lim_{\|}\xi\| \to \infty j(\xi) = \infty.$$

Consequently, j is coercive and the existence of a minimum is ensured. □

Example 3.0.4.
Consider the system with linearly dependent rows

$$x_1 + x_2 + x_3 + = 1,$$

$$2x_1 + 2x_2 + 2x_3 + = 2.$$

In this case n=2 and $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
Now,

$$AA^* = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}.$$

Therefore the critical points of the quadratic functional j given by (3.1) satisfy the equation:

$$AA^*\xi = b$$

i.e.,

$$3\xi_1 + 6\xi_2 = 1$$

$$6\xi_1 + 12\xi_2 = 2$$

$$\xi_1 = \frac{1 - 6\xi_2}{3}$$

i.e.,

$$= \frac{1}{3} - 2\xi_2.$$

So, there are infinitely many critical points given by

$$\xi = \begin{bmatrix} \frac{1}{3} - 2a \\ a \end{bmatrix}, \quad a \in R$$

Hence, a solution of the system is given by

$$x = A^*\xi = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} - 2a \\ a \end{bmatrix} = \begin{bmatrix} \frac{1}{3} - 2a + 2a \\ \frac{1}{3} - 2a + 2a \\ \frac{1}{3} - 2a + 2a \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

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